A Two-Country Case: Eaton-Kortum Model with General Equilibrium

Technology Changes and International Trade

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Preface

First and foremost, I would like to extend my gratitude to my supervisor, Alfonso Irarrazabal. Particularly important assistance was provided in the construction of models and in general economic theory.

This thesis does not only mean the end of my study as a master student in the University of Oslo, but also a treasure in my life. I would never forget what I have experienced during this studying in my left life. At this moment, I would like to thank all lecturers of the Department who make economics an exciting and interesting world. I also thank all of my friends in Beijing and Oslo, for the sharing of pleasure and desperation, for the emotional support and ardent caring, and for all those various memorable moments.

Finally, and most importantly I am also grateful to my parents' unconditional love. Without their encouragement and support from China I would not be here writing this acknowledgement.

I wish to point out that any errors or flaws in the thesis are entirely my responsibility.

Jia Zhao

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Summary

The aim of this paper is to analyze a two-country version of the Alvarez and Lucas (2007) model. In this model, each country has two production sectors with constant-return-to-scale: an intermediate goods sector and a final goods sector. Labor and intermediate goods are used as factors to produce both the final goods and intermediate goods. Production technology level of intermediate goods differs across goods when intermediate goods are at continuum. Dormbusch, Fischer and Samuelson (1977) pointed out the existence of variance in individual productivity. Intermediate goods can be considered as random variables drawn from a parameterized distribution. All the tradables are traded at the lowest prices. Following from Eaton-Kortum model, only the intermediate goods can be traded, and they are assumed to be tradable and continuum. The intermediate goods are traded at the lowest prices which include "iceberg" costs. The final goods are non-traded, consumption goods with a technology level common to all countries.

Here the technology level of intermediate goods production is given by the expression of both the absolute advantage and technology heterogeneity. And those random variables follow Frechet distribution with parameters λ and θ. λ mainly reflects the mean value of that distribution and θ mainly reflects the variance of that distribution. In economic, mean value stands by the absolute advantage of technology and variance stands by technology heterogeneity.

Within this framework, we can find out how the production technology level will affect international trade.

Firstly, I study how production technology level's change that is realized by changing λ and θ affects economic variables such as wage, GDP, country's total production in a simple, autarky economy. Under equilibrium, it has been found that all of the equilibrium prices are functions of wage rate in this economy, in the other word, all those prices (wage can be taken as the price of labor) are different multiples of the wage rate. Then I study what happens when λ and θ (technological absolute advantage and technology heterogeneity) change. When λ is larger than before, but θ is fixed, the conclusions I get are as follows. Wage rate level will be higher, total production including the final goods production and intermediate goods production is amplified. But on the contrary, the intermediate goods' price index (aggregate price)

decreases. The technology heterogeneity (mainly reflected by θ) plays very important role in this model, but how the change of θ affects variables in this economy can not be determined.

Secondly, I turn to an open economy with balance trade. And this open economy is the simple case which only has two countries (country i and country j) in the world with balanced trade. Then I characterize the equilibrium in an open economy. I assume that only country i gets technology improvement and then study the consequences of that. In this paper I only study the improvement of the productivity absolute advantage and omit the change of technological heterogeneity. Country i's GDP will increase along with the upgrading absolute advantage level. Going through technology improvement can enlarge the country i's trade volume compared to country j, when the intermediate goods production is more labor intensive than the final goods production, in other words, country i will experience exporting expansion relative to country j. At the same time, country i's wage level gets improved and would be higher than that of country j.

Contents

1 Induction

There exists different international trade models that explain the trade patterns and trade policies. For example, the standard Ricardian model and the Heckscher-Olhin model, which are called classical trade theories, assume that firms' productivities are homogeneous in one production sector. In reality, firms have large and persistent differences in terms of size, productivity. Bernard and Jensen (1995, 1999, 2001) started to research heterogeneity of firms.

Eaton and Kortum (2002) have proposed a new theory of international trade, which is based in the Ricardian model (based on the difference in absolute advantage of productivity and comparative advantage of productivity) with a continuum of goods and incorporates the geography role. Unlike the earlier theories, Eaton-Kortum (2002) model is competitive, but does not involve fixed costs and monopoly rents. Since fixed costs and monopoly rents present in reality, using a large body of general equilibrium is very helpful and easier to calibrate and analyze under competitive market conditions. Also papers following Eaton and Kortum' s framework adopt a realistic tractable parameterization of the firm's productivity heterogeneity. The productivity level can be described in two ways: one is the absolute advantage in productivity, the second is the technological heterogeneity.

Alvarez and Lucas (2007) studied the Eatom-Kortum (2002) model in a general equilibrium context. In Alvarez and Lucas (2007)'s model the production function is constant return to scale subjected to idiosyncratic productivity shocks. Under the assumption of perfect competition, buyers are driven by the lowest price, and trade would assign to the goods which are produced efficiently, subject to the lowest costs of transportation and other impediments.

The aim of this paper is to analyze a two-country version of the Alvarez and Lucas (2007) model. In this model, each country has two production sectors with constant-return-to-scale: an intermediate goods sector and a final goods sector. Labor and intermediate goods are used as factors to produce both the final goods and intermediate goods. Production technology level of intermediate goods differs across goods when intermediate goods are at continuum. Dormbusch, Fischer and Samuelson (1977) pointed out the existence of variance in individual productivity. Intermediate goods can be considered as random variables drawn from a parameterized distribution. All the tradables are traded at the lowest prices. Following from Eaton-Kortum model, only the intermediate goods can be traded, and they are assumed to be tradable and continuum. The intermediate goods are traded at the lowest prices which include "iceberg" costs. The final goods are non-traded consumption goods with a technology level common to all countries. Within this framework, we can find out how the production technology level will affect international trade.

I use the model to discuss what happens to economic variables such as wage, GDP, country's total production and trade volume, when the productivity level (absolute advantage and heterogeneity) changes both under the autarky economy and under the two-country case. In particular, I analyze how the results are affected by changing two technological parameters: θ and λ in this model.

The remainder of the paper is structured as follows.

In Section 2, I go through the Alvarez and Lucas (2007) model in the closed economy and solve for the general equilibrium. I solve for the equilibrium prices as functions of wage, in other words, all prices are different multiples of the wage rate (w). After that I analyze what happens to the variables such as the wage rate and equilibrium prices, as the results of productivity level changes. I give the analysis based on two different scenarios: λ changing and θ changing.

In Section 3, I present the open economy model from Alvarez and Lucas (2007), and simplify it down to only two countries. Then I define the equilibrium in such an open two-country economy. I assume that one country's absolute productivity advantage gets improved and analyze how this improvement affects both countries'GDP, trade volume, welfare and real consumption.

Conclusions are contained in Section 4.

2 The model: Closed Economy Equilibrium

In this section I present the closed economy equilibrium model and study the effects of productivity level changes.

The Eaton-Kortum (2002) model is Ricardian model with continuum goods under the constant-return technology. The new idea here in Alvarez and Lucas (2007) model is that there exists a two-parameter probabilistic model that generates the input requirements to produce each good. Of course this closed economy is organized in two sectors. One is the intermediate goods sector where continuum, differentiated goods are produced and the second is a unique, homogeneous final good sector. And then it is very helpful to introduce the general equilibrium model.

The good way is to begin with setting up the model in the simple context of a single, closed economy before turning to the study of a model of an open economy between two countries discussed in Section 3.

2.1 Preferences, Technology

In the simple,closed economy, both intermediate goods and final goods are produced. Both intermediate input and labor input are combined in the production of intermediate goods and final goods. I take the intermediate goods that are used to produce the intermediate goods as the intermediate input bundle. And the labor is taken as the only primary factor (non-produced) of production. The new idea mentioned above is that those two probabilistic parameters are labor and intermediate goods.

Suppose there are L units labor and all of them are also consumers in this closed economy. There is a unique, produced final good c, which is the only good that can be consumed by the consumers in this economy. Because c is the only good that can be valued by the consumers, c is also used as utility.

Throughout this paper I fellow Eaton-Kortum (2002) model's production technology.

Just as the same as in Alvarez and Lucas (2007), continuum intermediate goods are produced with the technology which affects production symmetrically via a Spence-Dixit-Stiglitz (SDS) aggregate. Since intermediate goods differ only in their costs and they are continuum, in this sense, it is convenient to name each intermediate good by its cost draw, $x_n > 0$,and to speak of "good x_n " This "good x_n"is produced in country n. In Eaton-Kortum (2002), x_n follows exponential distribution with parameter $\lambda : x \sim \exp(\lambda)$, the density function is $\phi(x)$.

Because Total Factor Productivity (TFP) levels vary across these intermediate goods. And the inverse of these TFP levels as random variables, independent across goods.

From Eaton-Kortum (2002) model, I assume this closed economy's production technology efficiency (Z_n) is a realization of an i.i.d which follows Frechet distribution[1].

From Recardian Model, with perfect competition we have:

$$\text{unit cost} = \text{price} = p_n = \frac{x_n}{z_n} \Rightarrow z_n = \frac{x_n}{p_n}$$

p_n: the price of good;

I can take the costs as the inverse of these TFP levels and then the costs can be looked as random variables, independent across goods, with a common density ϕ.And also we can say that $x^{-\theta}$ follows Frechet distribution, with the parameter λ and $\frac{1}{\theta}$[2].

2.2 Framework

In a closed economy, there are only 2 inputs during the production process: labor and intermediate goods (continuum), and also final goods c are produced and consumed by the consumers who are also the labor suppliers.

2.2.1 Final goods production

$$y = c = s_f^{\alpha} q_f^{1-\alpha} \tag{1}$$

Here the Cobb-Douglas production function is used, $0 < \alpha < 1$.

s_f is the labor that allocated to produce final goods.

q_f is the intermediate goods that are inputed to produce final goods.

Solve the problem of minimizing the cost of final goods to get the final goods's equilibrium prices[3]:

$$p = \alpha^{-\alpha} (1-\alpha)^{\alpha-1} w_f^{\alpha} P_m^{1-\alpha} \tag{2}$$

w_f is the wage rate in this closed economy.

P_m is the price index of intermediate goods.

Under equilibrium condition the price of one good shoule be equal to the marginal cost of that good. So we can say this is the equilibrium price of final goods under closed economy.

[1]More details such as mean, variance of Frechet distribution have been shown in Appendix 5.2.
[2]The derivation details of $\frac{1}{\theta}$ is given in the Appendix 5.1
[3]The procedure of soloving the problem is given in Appendix 5.3

2.2.2 Intermediate goods production

We have mentioned above that the intermediate goods are continuum and differentiated. So let $q\left(x\right)$ be the production function of intermediate good x. And we already know $x \sim \exp\left(\lambda\right)$, with the density $\phi\left(x\right)$.

The production function:

$$q(x) = \left[\int_0^\infty q_m(x)^{1-\frac{1}{\eta}}\phi(x)dx\right]^{\frac{\eta}{\eta-1}} \tag{3}$$

Note: η is the elasticity among intermediate goods and these differentiated goods are imperfect substitutes from the perspective of the buyers ,so $\eta > 0$ is assumed.

The labor to produce intermediates is $s(x)$, $x > 0$,then the labor used to produce intermediate goods is $\int_0^\infty s\left(x\right)\phi\left(x\right)dx$.

The total labor supply of the closed economy is:

$$s_f + \int_0^\infty s\left(x\right)\phi\left(x\right)dx = 1 \tag{4}$$

The total production function is :

$$q_f + \int_0^\infty q_m(x)\phi(x)dx = q \tag{5}$$

and

$$q = \left[\int_0^\infty q(x)^{1-\frac{1}{\eta}}\phi(x)dx\right]^{\frac{\eta}{\eta-1}} \tag{6}$$

Then the intermediate goods producer $x's$ production function is :

$$q\left(x\right) = x^{-\theta}s\left(x\right)^{\beta}q_m\left(x\right)^{1-\beta} \tag{7}$$

$q_m\left(x\right)$ is looked as the intermediate input bundle to produce intermediate goods;

$s\left(x\right)$ is the labor that are allocate to produce intermediate goods;

Then I try to find the intermediate goods' equilibrium price index P_m.What's more, in this closed economy any producers are free to produce any special good with the same acess to the same production technology, the technology parameter should be stochastic $x^{-\theta}$.

After minimizing intermediate goods production costs, we can get the equilibrium price of intermediate goods[4]:

$$p(x) = \beta^{-\beta}(1-\beta)^{\beta-1} w^{-\beta} P_m^{1-\beta} x^{\theta} \tag{8}$$

w is the wage rate, in the same economy we can take it as granted that $w = w_f$.
And so the intermediate goods' price index is[5]:

$$P_m = \left[\lambda \int_0^{\infty} e^{-\lambda x} p(x)^{1-\eta} dx\right]^{\frac{1}{1-\eta}} \tag{9}$$

Putting (8) into (9) and get the intermediate goods' price index:

$$P_m = \lambda \left\{\int_0^{\infty} e^{-\lambda x}\left[Bw^{\beta} P_m^{1-\beta} x^{\theta}\right]^{1-\eta} dx\right\}^{\frac{1}{1-\eta}} \tag{10}$$

Now we do some transfers to let the formula look better to analyze.
Let $z = \lambda x$,then $\frac{dz}{dx} = \lambda \Rightarrow dz = \lambda dx$ and $e^{-z} = e^{-\lambda x}$.
The transfered function is: (10)

$$\begin{aligned} P_m &= Bw^{\beta} p_m^{1-\beta}\left[\lambda^{-\theta(1-\eta)} \int_0^{\infty} e^{-\lambda x}(\lambda x)^{\theta(1-\eta)} dz\right]^{\frac{1}{1-\eta}} \\ &= Bw^{\beta} p_m^{1-\beta} \lambda^{-\theta}\left[\int_0^{\infty} e^{-\lambda x}(\lambda x)^{\theta(1-\eta)} dz\right]^{\frac{1}{1-\eta}} \end{aligned}$$

where there is Gamma function $\Gamma(\xi)$, evaluated at the argument $\xi = 1 + \theta(1-\eta)$, convergence of the integral requires $1 + \theta(1-\eta) > 0$.

$$A(\theta, \eta) = \left[\int_0^{\infty} e^{-z} z^{\theta(1-\eta)} dz\right]^{\frac{1}{1-\eta}}$$

So the intermediate goods' price index is written as:

$$\begin{aligned} P_m &= Bw^{\beta} P_m^{1-\beta} \lambda^{-\theta} A(\theta, \eta) \\ &\Rightarrow P_m = [A(\theta, \eta)B]^{\frac{1}{\beta}} w \lambda^{-\frac{\theta}{\beta}} \end{aligned} \tag{11}$$

This is the intermediate goods' price index under minimum cost.

[4]The procedure of soloving minimizing cost problem is shown in Appendix 5.4
[5]This formula has been proofed by Eaton-Kortum <Technology in the Global Economy:A Framework for Quantitative Analysis> (2005). page25-page50.

Put (11) into (2) : the equilibrium price of individual final goods:

$$p = \alpha^{-\alpha} (1-\alpha)^{\alpha-1} \lambda^{\frac{-\theta(1-\alpha)}{\beta}} (AB)^{\frac{1-\alpha}{\beta}} w \tag{12}$$

.

Put (11) into (8), the equilibrium pice of this closed economy's individual intermediate goods is:

$$p(x) = A^{\frac{1-\beta}{\beta}} B^{\frac{1}{\beta}} x^{\theta} \lambda^{\frac{-\theta(1-\beta)}{\beta}} w \tag{13}$$

It can be seen that all of prices, p, P_m and $p(x)$ can be written as the multiples of the wage rate w, technology parameters λ, θ and the Cobb-Douglas costant share parameter α, β (B is function of β[6]). Given the equilibrium prices, it is easy to solve the equilibrium quantities by using the familiar Cobb-Douglas constant share formulas. Now we turn to study how the technology parameters (λ, θ) change to affect the economy.

2.2.3 Equilibrium

For easier, let the final goods' price to be nomolized : $p = 1$ and let the labor supply in this economy be $L = 1$.

Using equation (12) to solve for wages w:

$$w = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}} \tag{14}$$

Now the wage can be expressed in the function of parameters: $\alpha, A, B, \lambda, \theta$.

Final goods Since total income of the consumers is wL, and the total income is equal to the total expenditure which is py, then there is:

The totoal final goods production:

$$\begin{aligned} py &= wL \\ &\Rightarrow y = w = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(1-\alpha)}{\beta}} (AB) \end{aligned} \tag{15}$$

The final goods production function(1): $y = s_f^{\alpha} q_f^{1-\alpha}$, so the labor allocated in the final goods production is:

[6] The details are given in Appendix 5.4

$$s_f = \left[\left(\frac{\alpha}{1-\alpha} \right) (AB)^{\frac{1}{\beta}} \lambda^{-\frac{\theta}{\beta}} \right]^{1-\alpha} w = \alpha \tag{16}$$

Obviously, the labor allocated to produce final goods is equal to the Cobb-Douglas production function parameter α. So this part of labor is only related with α.

Intermediate goods From (13), the individual price of intermediate goods is:

$$p(x) = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(\beta-\alpha)}{\beta}} (AB)^{\frac{\alpha}{\beta}} A^{-1} \tag{17}$$

The intermediate goods' price index is:

$$P_m = \alpha^{\alpha} (1-\alpha)^{1-\alpha} (AB)^{\frac{\alpha}{\beta}} \lambda^{-\frac{\theta\alpha}{\beta}} \tag{18}$$

From (4), the labor allocated to produce intermediate goods is :

$$\int_0^{\infty} s(x)\, \phi(x)\, dx = 1 - s_f = 1 - \alpha \tag{19}$$

The intermediate goods to produce final goods is:

$$\begin{aligned} q_f &= \left[\frac{\alpha}{1-\alpha} (AB)^{\frac{1}{\beta}} \lambda^{-\frac{\theta}{\beta}} \right]^{-\alpha} w \\ &= (1-\alpha) \lambda^{\frac{\theta}{\beta}} (AB)^{-\frac{1}{\beta}} \end{aligned} \tag{20}$$

Now we need to find the intermediate goods that are used to procuce intermediate goods as input bundle.

From (7) we can get the rate of intermediate input bundle over labor used to produce intermediate:

$$\frac{s(x)}{q_m(x)} = \left[\left(\frac{1-\beta}{\beta} \right) (AB)^{-\frac{1}{\beta}} \lambda^{\frac{\theta}{\beta}} \right]^{-1} \tag{21}$$

$$q_m(x) = (1-\alpha) \left[\left(\frac{1-\beta}{\beta} \right) (AB)^{-\frac{1}{\beta}} \lambda^{\frac{\theta}{\beta}} \right] \tag{22}$$

Then we will have :

$$\int_0^{\infty} q_m(x)\, \phi(x)\, dx = (1-\alpha) \left(\frac{1-\beta}{\beta} \right) (AB)^{-\frac{1}{\beta}} \lambda^{\frac{\theta}{\beta}} \tag{23}$$

And (5) tells : $q_f + \int_0^{\infty} q_m(x)\phi(x)dx = q$,

$$q = (1-\alpha)\left(\frac{1-\beta}{\beta}\right)(AB)^{-\frac{1}{\beta}}\lambda^{\frac{\theta}{\beta}} + (1-\alpha)\lambda^{\frac{\theta}{\beta}}(AB)^{-\frac{1}{\beta}}$$

The total intermediate goods production in the economy is:

$$q = (AB)^{-\frac{1}{\beta}}\lambda^{\frac{\theta}{\beta}}\left(\frac{1-\alpha}{\beta}\right) \tag{24}$$

2.3 The effect of technology level changes

In this model the productivity level involves both the absolute advantage and heterogeneity. Both sides play important roles in the trade in the view of technology. From Eaton-Kortum (2002) model, the productivity levels follow Frechet distribution with the parameters λ and $\frac{1}{\theta}$. And Frechet distribution's mean and variance are decided by θ and λ[7]. And the fact is that λ does mainly influence to the mean value of the distribution and θ does mainly influence on variance[8].

Now I analyze if productivity level changes (λ and $\frac{1}{\theta}$ change) what will happen on this production structure. I carry out this work by analyzing λ and θ seperately: λ changes with fixed θ and θ changes with fixed λ. I will not discuss what happens when both θ and λ change at the same time.

Note: (1) $\lambda > 1$.

(2) α and β are Cobb-Douglas production function parameters and are constants. And

in this model $B = \beta^{-\beta}(1-\beta)^{\beta-1}$,then B is of course constant.

(3) The parameter A, as we have know from above, $A(\theta,\eta) = \Gamma(\xi)$. It is a Gamma function with

the parameter $\xi, \xi = 1 + \theta(1-\eta)$.

(4) The allocation of labor and materials between the two sectors is independent of the value η,

η is the elasticity among intermediate goods and $\eta > 0$.

2.3.1 When θ is fixed, λ changes

We already have known that change of λ will mainly cause the substantial change of the mean value of the random variables draw. In this analysis, I want to find the relationship between λ and the trade, economical variables. When λ becomes bigger, that means

[7] The proofment is shown in Appendix 5.1

[8] See Appendix 5.2

the productivity absolute advantage become larger, what will happen on the trade and economical variables in this model?

Effects on wage and price: The wage rate (14)

$$w = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}},$$

It is obvious to find w is increasing in λ. When λ increases (decreases), the wage level w will correspondingly increases (decreases). Accordingly the production of final goods y will increases (decreases) when λ increases (decreases)[9].

The total labor supply(16) in this economy will not change. Also the both parts allocated over final goods and intermediate goods will not chage. Because they are determined by α and here α is a constant.

The individual price of intermediate goods (17) is:

$$p(x) = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(\beta-\alpha)}{\beta}} (AB)^{\frac{\alpha}{\beta}} A^{-1}$$

This function is a little complicated because of the existance of α and β.I discuss this case from the following two sides[10].

(1) When $\alpha > \beta$, then $\beta - \alpha > 0$. Then $p(x)$ is an increasing function in λ.

(2) When $\alpha < \beta$, then $\beta - \alpha < 0$. Then $p(x)$ is an decreasing function in λ.

Here α and β play pivotal roles. If the final good production is more labor intensive, $p(x)$ progressively increases in λ.If the intermediate good production is more labor intensive than the final good, $p(x)$ progressively decreases in λ.

It has be given that (18), the intermediate goods' price index:

$$P_m = \alpha^{\alpha} (1-\alpha)^{1-\alpha} (AB)^{\frac{\alpha}{\beta}} \lambda^{-\frac{\theta\alpha}{\beta}}$$

Obviously, when λ increases (decreases) P_m will decrease (increase). α and β do nothing here[11].

Because of (2)

$$p = \alpha^{-\alpha} (1-\alpha)^{\alpha-1} w_f^{\alpha} P_m^{1-\alpha}$$

The final good's price is positive correlated with the price index of intermediate good, so

[9] The proofment is in Appendix 5.6.1

[10] The proofs are given in Appendix 5.6.2

[11] The proofment is given in Appendix 5.6.4

it will get decreased with larger λ[12].

Effects on productions: The intermediate goods used to produce final goods is:

$$q_f = (1-\alpha)\,\lambda^{\frac{\theta}{\beta}}\,(AB)^{-\frac{1}{\beta}}$$

which is also an increasing function in λ[13].

The intermediate input bundle to produce intermediate goods is

$$q_m(x) = (1-\alpha)\left[\left(\frac{1-\beta}{\beta}\right)(AB)^{-\frac{1}{\beta}}\,\lambda^{\frac{\theta}{\beta}}\right]$$

obviously it is an increasing function in λ[14].

Total production of intermediate is

$$q = (AB)^{-\frac{1}{\beta}}\,\lambda^{\frac{\theta}{\beta}\left(\frac{1-\alpha}{\beta}\right)}$$

if λ increases (decreases) q will increases (decreases)[15].

Then I can get the conclusions:

If λ gets improvement with fixed θ, the intermediate goods' production will be amplified and then the part of intermediate good that is used to produce final good will increase. As the sequenec, final good production also gets enlarged. So the result would be the total good production will increase. Labor input is constant in this model. Higher λ meaning bigger absolute advantage, always leads to improve the production (including intermediate goods and final goods) and wage level, but reduces the intermediate goods price index and final goods price. Reduction of intermediate goods price index causes the price inequality among countries in the world. This unbalance in price is the main factor to make trade take place.

2.3.2 When λ is fixed , θ changes

In this model, θ mainly affects the variance of the productivity level distribution, θ's change will cause technological heterogeneity change. The technological heterogeneity is positively related with parameter θ.And θ always shows up as a power number and an argument of Gamma Function. So it is a very sensitive parameter and of course it will be a little complicated to analyze.

[12]You can find the evidence in Appendix 5.6.4

[13]It has been shown in Appendix 5.6.5

[14]The proofment is given in Appendix 5.6.6

[15]The proofment is given in Appendix 5.6.7

Wage rate is $w = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}}$, as we have talked above, here A is some kind of Gamma function of $1 + \theta (1-\eta) > 1$, and how A changes with increasing θ is uncertain. And I have mentioned before that λ is bigger than 1 and $0 < \alpha < 1$. So when θ is larger than before, the part of $\lambda^{\frac{\theta(1-\alpha)}{\beta}}$ will increase. There is an ambiguous relationship between θ and wage rate w.

Since $y = w$, the total consumption y has the same condition as w.

The individual price of intermediate goods

$$p(x) = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(\beta-\alpha)}{\beta}} (AB)^{\frac{\alpha}{\beta}} A^{-1}$$

The intermediate goods' price index:

$$P_m = \alpha^{\alpha} (1-\alpha)^{1-\alpha} (AB)^{\frac{\alpha}{\beta}} \lambda^{-\frac{\theta\alpha}{\beta}}$$

The intermediate goods to produce final goods:

$$q_f = (1-\alpha) \lambda^{\frac{\theta}{\beta}} (AB)^{-\frac{1}{\beta}}$$

The intermediate input bundle to produce intermediate goods $q_m(x) = (1-\alpha) \left[\left(\frac{1-\beta}{\beta} \right) (AB)^{-\frac{1}{\beta}} \lambda^{\frac{\theta}{\beta}} \right]$ and total production of intermediate:

$$q = (AB)^{-\frac{1}{\beta}} \lambda^{\frac{\theta}{\beta}} \left(\frac{1-\alpha}{\beta} \right)$$

All of them have the part of A (some kind of Gamma Function), so the chage of θ causes ambiguous changes on those variables.

All of functions talked above have the uncertain part $(AB)^{-\frac{1}{\beta}}$ and the increasing part $\lambda^{\frac{\theta}{\beta}}$ with increasing θ. This means that when θ changes,but λ is fixed, both increasing and decreasing may appear in all those variables. It is difficult to get the explicit relationship among them. In this model, the relationship between θ's change and other variables such as wage, intermediate good price index can not be determined.

2.4 Welfare

Naturaly, I use real wage to describe the welfare level of this closed economy.

The real wage here is :

$$\frac{w}{p} = w = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}}$$

Here p is normalized to 1. It has shown that w is an increasing function in λ with fixed θ, so I can conclude that the welfare level (real wage) is positive correlated with λ. But I can not get clear relationship between w and θ with fixed λ. When θ changes with fixed λ, it is difficult to get the conclusion of welfare level change.

In summary, the welfare will get improvement if the productivity absolute advantage become larger, but there is no clear relationship when the technological heterogeneity gets notable.

Under equilibrium, the productivity levels which depends on the parameters λ and θ do lots of effects on all prices (wage can be looked as the labor's price).

3 The Open Economy: Two-Country Case

Now it is time to turn to a small open economy which contains two countries i, j. Both countries have the production condition which is described in Section 2. And those two countries can trade each other. But note here that only intermediate goods can be traded between those two countries. In this section I discuss the equilibrium and try to find out the consequences of productivity level changing in one country.

3.1 Framework

Let the total labor endowments be $L = (L_i, L_j)$,here L_i is the total efficiency units of labor in country i (or country j) and it is not mobile between countries. The exponential distributions that define each country's productivity have the parameters $\lambda = (\lambda_i, \lambda_j)$. But the other productivity parameter θ is identical in both countries. Country i and country j have same preferences α, β and η. Each of those two countries' production structures is as same as described in Section 2. And country i and country j can trade the intermediate goods with each other. In this model, only the intermediate goods can be traded between two countries subject to transportation costs.

Transportation cost is τ_{ij}, it means the cost that country j transports one unit of any intermediate goods to country i.And it is obvious that τ_{ij} is proportional to distance. Natural to assume that $0 < \tau_{ij} \leq 1$, with equality if $i = j, \tau_{ij} = \tau_{ji}$ for all i, j.

Section 2 has exploited the assumptions of perfect competition and constant returns to solve for all equilibrium prices as multiples of wage w, with productivity coefficients. Follows the same procedure, here I can apply it to the two-country case to calculate equilibrium quantities.

Let $X = (X_i, X_j)$ be the vector of productivity draws for any given tradeable goods for the two countries, and $x \in R^n_+$.

Assume that those draws are independent across countries and $X_i \sim \exp(\lambda_i)$, so that the joint density of x is:

$$\phi(x) = \lambda_1 \lambda_2 e^{-(\lambda_i x_i + \lambda_j x_j)}$$

The same as the closed economy, we let $q_i(x)$ be country i's consumption of tradeable good x.

q_i is the aggregated consumption of country i :

$$q_i = \left[\int q_i(x)^{1-\frac{1}{\eta}} \phi(x)\, dx \right]^{\frac{\eta}{\eta-1}} \tag{25}$$

Then we can get the intermediate goods price index of country i is :

$$P_{mi} = \left[\int p_i(x)^{1-\eta} \phi(x)\, dx \right]^{\frac{1}{\eta-1}} \tag{26}$$

Note: η is the elasticity among intermediate goods, and $0 < \eta < 1$.

3.2 Market Price

From (11), the intermediate goods' individual tradable prices avaiable in the market are :

$$(p_i(x), p_j(x)) = \left(B x_i^{\theta} w_i^{\beta} P_{mi}^{1-\beta} \frac{1}{\tau_{ji}}, B x_j^{\theta} w_j^{\beta} P_{mj}^{1-\beta} \frac{1}{\tau_{ij}} \right)$$

According to the asummption, only the intermediate good with lower price can be traded by these two countries, so the tradeable price should be:

$$\begin{aligned} p_i(x) &= \min \left[B x_i^{\theta} w_i^{\beta} P_{mi}^{1-\beta} \frac{1}{\tau_{ji}}, B x_j^{\theta} w_j^{\beta} P_{mj}^{1-\beta} \frac{1}{\tau_{ij}} \right] \\ &\Rightarrow\ p(x)^{\frac{1}{\theta}} = \min B^{\frac{1}{\theta}} \left[x_i^{\theta} w_i^{\beta} P_{mi}^{1-\beta} \frac{1}{\tau_{ji}}, x_j^{\theta} w_j^{\beta} P_{mj}^{1-\beta} \frac{1}{\tau_{ij}} \right]^{\frac{1}{\theta}} \end{aligned}$$

The price index that can be taken by the consumers is:

$$P_{mi} = \left[\int p_i(x)^{1-\eta} \phi(x)\, dx \right]^{\frac{1}{\eta-1}} \Longrightarrow P_{mi}^{\eta-1} = \int p_i(x)^{1-\eta} \phi(x)\, dx \tag{27}$$

Here I introduce some mathematics properties of the exponational distribution to help to analyze:

Proposition 1 *(1)* $x \backsim \exp(\lambda)$ *and* $k > 0 \Longrightarrow kx \backsim \exp\left(\frac{\lambda}{k}\right)$;

Proposition 2 *(2) If* x *and* y *are independent* , $x \backsim \exp(\lambda), y \backsim \exp(\mu)$, *then* $z = \min(x, y) \Longrightarrow z \backsim \exp(\lambda + \mu)$

So I apply those properties to let $z = p_i(x)^{\frac{1}{\theta}} = x\left[w^{\frac{\beta}{\theta}} P_m^{\frac{1-\beta}{\theta}} \left(\frac{1}{\tau}\right)^{\frac{1}{\theta}}\right]$, and $x \sim \exp(\lambda)$, here let $k = \left[w^{\frac{\beta}{\theta}} P_m^{\frac{1-\beta}{\theta}} \left(\frac{1}{\tau}\right)^{\frac{1}{\theta}}\right] > 0$.

By using the property (1), there is the pdf of variable z:

$$f(z) = \psi(x) = \frac{\lambda}{\left(\frac{w^\beta P_m^{1-\beta}}{\tau}\right)^{\frac{1}{\theta}}} = \left(\frac{w^\beta P_m^{1-\beta}}{\tau}\right)^{-\frac{1}{\theta}} \lambda$$

this is the exponentially distributed parameter of z subject to x.

By using the property (2) we can get:

$$\psi_{ji} + \psi_{ij} = \left(\frac{w_i^\beta P_{mi}^{1-\beta}}{\tau_{ji}}\right)^{-\frac{1}{\theta}} \lambda_i + \left(\frac{w_j^\beta P_{mj}^{1-\beta}}{\tau_{ij}}\right)^{-\frac{1}{\theta}} \lambda_j \tag{28}$$

Then "$\psi_{ji} + \psi_{ij}$" is the parameter of the $\min\{z\}$'s distribution.

Now I try to find out the price index of the tradeables.

Let $\mu = B^{-\frac{1}{\theta}}\psi$, and $u = p_i(x)^{\frac{1}{\theta}}$ so $u \backsim \exp(\mu)$. Then put them into (26), we will get:

$$P_m^{1-\eta} = \int_0^\infty u^{\theta(1-\eta)} \mu e^{-\mu u} du = \mu \int_0^\infty u^{\theta(1-\eta)} e^{-\mu u} du$$

To simplify the function, I let $r = \mu u$,then $u = \frac{r}{\mu} \Longrightarrow u^{\theta(1-\eta)} = r^{\theta(1-\eta)} \mu^{-\theta(1-\eta)}$ and $\frac{dr}{du} = \mu \Rightarrow du = \frac{dr}{\mu}$

Then there has:

$$\begin{aligned} P_m^{1-\eta} &= \left[\mu \int_0^\infty r^{\theta(1-\eta)} \mu^{-\theta(1-\eta)-1} e^{-r} dr\right]^{1-\eta} \\ &= \left[\int_0^\infty r^{\theta(1-\eta)} \mu^{-\theta(1-\eta)} e^{-r} dr\right]^{1-\eta} \\ &= \left[\mu^{-\theta(1-\eta)} \int_0^\infty r^{\theta(1-\eta)} e^{-r} dr\right]^{1-\eta} \end{aligned}$$

The same as in closed economy, here is the Gamma function: $\int_0^\infty r^{\theta(1-\eta)} e^{-r} dr = A(\theta, \eta)$, so we get

$$P_m^{1-\eta} = \mu^{-\theta(1-\eta)} A^{1-\eta} \text{ and } \mu = B^{-\frac{1}{\theta}} \left(\psi_{ji} + \psi_{ij}\right)$$

$$P_m(x) = AB(\psi_{ii}+\psi_{ij})^{-\theta} \quad (29)$$
$$= AB\left[\left(\frac{w_i^{\beta}P_{mi}^{1-\beta}}{\tau_{ji}}\right)^{-\frac{1}{\theta}}\lambda_i + \left(\frac{w_j^{\beta}P_{mj}^{1-\beta}}{\tau_{ij}}\right)^{-\frac{1}{\theta}}\lambda_j\right]^{-\theta} \quad (30)$$

$P_m(x)$ is the tradable goods price index in the world market, which includes country $i's$ market and country $j's$ market.

It is obvious that $P_m(x)$ is the function of P_{mi}, P_{mj} and the wages of thoes two countries w_i, w_j. In section 2, I have shown that the intermediate good price index is also the function of wage. So using this 30 we can solve for $P_m(x)$ as the function of wage vector $w(w_i, w_j)$, so we can get the conclusion that the price index of tradeable intermediate goods is a function of the wage vector. We can write $P_m(x)$ as $P_m(w)$. On the other hand, if we want our price to be more competitive in this world market, we can modify the wage and the productivity level to get the lower price.

3.3 The Expenditure Share for Each Country

Assume D_{ij} is the fraction of country i's per capita spending $P_{mi}q_i$ on tradeables that is spent on goods from abroad. So the total spending in i on goods from abroad is:

$$P_{mi}q_iD_{ij} = \int_{B_{ij}} p_i(x)\,q_i(x)\,\phi(x)\,dx$$

here $B_{ij} \subset R_+^n$ is the set on which j attains the minimum in price vector.

So we can get $D_{ij} = \frac{\int_{B_{ij}} p_i(x)q_i(x)\phi(x)dx}{P_{mi}q_i}$ and we can see that D_{ij} is relative to $\int_{B_{ij}} p_i(x)\,\phi(x)\,dx$.

If country i imports intermediate goods from country j, country j's intermediate good price including transpotation cost should be lower than country i's intermediate good price. It means:

$$x_j^{\theta}w_j^{\beta}P_{mj}^{1-\beta}\frac{1}{\tau_{ij}} \leq x_i^{\theta}w_i^{\beta}P_{mi}^{1-\beta}\frac{1}{\tau_{ii}}$$

Note: $\tau_{ii} = 1$,then $\frac{1}{\tau_{ii}} = 1$.

There is another mathamatics property of the exponential distribution.

Proposition 3 *when x and y are indepedent and* $x \backsim \exp(\lambda), y \backsim \exp(\mu) \Longrightarrow \Pr\{x \leq y\} = \frac{\lambda}{\lambda+\mu}$

We know that $x_j\left(w_j^{\beta}P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}\right)^{\frac{1}{\theta}}$ is exponential with parameter ψ_{ij} ,and $x_i\left(w_i^{\beta}P_{mi}^{1-\beta}\frac{1}{\tau_{ii}}\right)^{\frac{1}{\theta}\prime}$s exponential with parameter ψ_{ii}, here

$$\psi_{ii} = \left(w_i^{\beta} P_{mi}^{1-\beta} \frac{1}{\tau_{ii}} \right)^{\frac{1}{\theta}}$$

So I can get the share that country i' expenditure on importing goods which produced in country j is the probability that for a particular good x, the lower price vendors for buyers in i are producers in j. Using the property, this probability can be calculated directly:

$$D_{ij} = \Pr \left\{ x_j^{\theta} w_j^{\beta} P_{mj}^{1-\beta} \frac{1}{\tau_{ij}} \leq x_i^{\theta} w_i^{\beta} P_{mi}^{1-\beta} \frac{1}{\tau_{ii}} \right\}$$

$$D_{ij} = \frac{\psi_{ij}}{\psi_{ii} + \psi_{ij}} \tag{31}$$

$$D_{ji} = \frac{\psi_{ji}}{\psi_{ji} + \psi_{jj}} \tag{32}$$

Note that: $D_{ij} + D_{ii} = 1$ and $D_{ji} + D_{jj} = 1$.

3.4 Trade Balance

The payments for tradeables flowing into i from abroad must equal the payments flowing out of i to the foreign country.

Expenditure of country i including both transporttation costs is :

$$L_i P_{mi} q_i D_{ij}$$

The amount of buyers in j spend on tradeables from i:

$$L_j P_{mj} q_j D_{ji}$$

So there exists balance:

$$L_i p_{mi} q_i D_{ij} = L_j p_{mj} q_j D_{ji} \tag{33}$$

Under equilibium, payments to foreigner equals receipts from foreigner.

As I mentioned before, all prices in this world can be expressed in terms of wages. And D_{ij} and D_{ji} (31and32) can be solved for the prices $P_m = (P_{mi}, P_{mj})$, so I can conclude that D_{ij} and D_{ji} can be expressed as a function of wage vector. The impact of the country j on the behavior of individual producers in i is entirely determined by the price index

P_{mi}.

The trade banlance condition is:

$$L_i p_{mi} q_i D_{ij} = L_j p_{mj} q_j D_{ji}$$

can be viewed as an equation in wage vector w and the vector q of tradeables consumption per capital.

Now I first consider the trade balance in country i, using (33)

$$L_i P_{mi} q_i = L_i p_{mi} q_i D_{ij} + L_i P_{mi} q_i D_{ii} \tag{34}$$

There would also be the same condition in country j.

Absence of indirect taxes, GDP equals to national income:

$$L_i p_i c_i = L_i w_i \text{ here } c_i \text{ is the consumption of country } i$$

Because of the Cobb-Douglas constant share formulas in final goods production:

$$(1-\alpha) L_i p_i c_i = (1-\alpha) L_i w_i = L_i P_{mi} q_{fi} \Rightarrow 1-\alpha = \frac{P_{mi} q_{fi}}{w_i} \tag{35}$$

As the same in final goods, the intermediate goods production is:

$$1-\beta = \frac{L_i P_{mi}(q_i - q_{fi})}{L_i P_{mi} q_i} = \frac{q_i - q_{fi}}{q_i} \tag{36}$$

Together (35) and (36) implying:

$$\begin{aligned}
(1-\alpha) L_i w_i &= L_i P_{mi} q_{fi} \\
(1-\beta) L_i P_{mi} q_i &= L_i P_{mi} q_i - L_i P_{mi} q_{fi} \\
\beta L_i P_{mi} q_i &= L_i P_{mi} q_{fi} \Rightarrow \frac{q_{fi}}{q_i} = \beta \\
\beta L_i P_{mi} q_i &= (1-\alpha) L_i w_i
\end{aligned}$$

And using (34):

$$\beta \left(L_j P_{mj} q_j D_{ij} + L_i P_{mi} q_i D_{ii} \right) = (1-\alpha) L_i w_i \tag{37}$$

$$\frac{\beta}{1-\alpha} \left(L_j P_{mj} q_j D_{ij} + L_i P_{mi} q_i D_{ii} \right) = L_i w_i = L_i p_i c_i \tag{38}$$

So I can get:

$$L_i P_{mi} q_i = \frac{1-\alpha}{\beta} L_i w_i \tag{39}$$

Symmetrically, I have the same in country j : $L_j P_{mj} q_j = \frac{(1-\alpha)}{\beta} L_j w_j$

Using (36) and (39)

$$\frac{\beta}{(1-\alpha)} \left(L_j P_{mj} q_j D_{ji} + L_i P_{mi} q_i D_{ii} \right) = L_i w_i \tag{40}$$

And then using the trade balance (34), it is obvious:

$$\frac{\beta}{(1-\alpha)} \left[\frac{(1-\alpha)}{\beta} L_j w_j + \frac{1-\alpha}{\beta} L_i w_i \right] = L_i w_i$$

Then in both countries:

$$\begin{aligned} L_j w_j D_{ji} + L_i w_i D_{ii} &= L_i w_i \\ L_j w_j D_{jj} + L_i w_i D_{ji} &= L_j w_j \end{aligned}$$

Using this equation I can solve the exact wage vector w to get the trade balance.

3.4.1 GDP and Trade Volume

For easier, I just analyze country i' s condition to get the information that we want.

Country i 's GDP is:

$$L_i p_i c_i$$

The labor income of country i is:

$$L_i w_i$$

In the final goods market, the value-added is:

$$L_i p_i c_i - L_i P_{mi} q_{fi}$$

And the income of labor that is allocated in final goods' production is:

$$L_i w_i s_{fi}$$

Value-added in intermediate goods which are traded with country j is:

$$\left(L_j P_{mj} q_j D_{ji} + L_i P_{mi} q_i D_{ii} \right) - L_i P_{mi} q_{mi}$$

And the income of lavbor that is allocated in intermediate goods' production is:

$$L_i w_i (1 - s_{fi})$$

When country i imports goods from country j, the value-added is:

$$L_i P_{mi} q_i - L_i p_{mi} q_i \left(D_{ij} + D_{ii}\right)$$

It is obvious that country j should have the same economical variables.

3.4.2 Equilibrium Quantities

The national GDP of country i is:

$$GDP = L_i w_i = \alpha^\alpha \left(1-\alpha\right)^{1-\alpha} \lambda_i^{\frac{\theta(1-\alpha)}{\beta}} \left(AB\right)^{\frac{\alpha-1}{\beta}}$$

Importing value of country i is :

$$\begin{aligned} & L_i P_{mi} q_i (1 - D_{ii}) \\ = \quad & L_i P_{mi} q_i D_{ij} = \alpha^\alpha \left(1-\alpha\right)^{1-\alpha} \lambda_i^{\frac{\theta(1-\alpha)}{\beta}} \left(AB\right)^{\frac{\alpha-1}{\beta}} \left(\frac{1-\alpha}{\beta}\right) D_{ij} \end{aligned}$$

The volume of trade of country i :

$$\begin{aligned} \frac{\text{Importing value}}{GDP} &= v_i = \frac{L_i P_{mi} q_i (1 - D_{ii})}{L_i w_i} \\ &= \frac{P_{mi} q_i (1 - D_{ii})}{w_i} \\ &= \frac{1-\alpha}{\beta}(1 - D_{ii}) \\ &= \frac{1-\alpha}{\beta} D_{ij} \end{aligned}$$

Real consumption:

$$\frac{GDP}{L_i p_i c_i} = \frac{L_i w_i}{L_i p_i c_i} = \frac{w_i}{p_i c_i} = A\left(AB\right)^{-\frac{\alpha}{\beta}} \lambda_i^{\frac{\theta(\alpha-\beta)}{\beta}} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}$$

3.4.3 Excess Labor Demand

From (**??**), s_{fi} is labor's share in the production of final goods and F_i is the fraction of country i's spending on tradeables that the producers of country i can get in this world market.

$$s_{fi} = \alpha$$

Both s_{fi} and F_i are functions of w. If there exists a wage vector , both of them can be solved.

Let excess labor demand be $Z_i(w)$ .Then we have :

$$\begin{aligned}
Z_i(w) w_i + L_i w_i \left[1 - s_{fi}(w)\right] &= \sum_{j=i,j} L_j w_j (1-\alpha) D_{ji}(w) \\
Z_i(w) w_i + L_i w_i (1-\alpha) &= L_j w_j (1-\alpha) D_{ji}(w) + L_i w_i (1-\alpha) D_{ii}(w)
\end{aligned}$$

Now we view solving these equations as finding the zeros of an excess demand system:

$$Z(w) = \frac{1}{w_i}\left[\sum_{j=i,j} L_j w_j (1-\alpha) D_{ji}(w) - L_i w_i (1-\alpha)\right] \tag{41}$$

and (41) can be transfered:

$$Z(w) = L_j \frac{w_j}{w_i}(1-\alpha) D_{ji}(w) + (1-\alpha)\left[1 - D_{ii}(w)\right] L_i$$

Because $D_{ij} + D_{ii} = 1$, so we have $1 - D_{ii} = D_{ij}$, then:

$$\begin{aligned}
Z(w) &= \frac{1}{w_i}\left[L_j w_j (1-\alpha) D_{ji}(w) + (1-\alpha) D_{ij} L_i w_i\right] \\
&= \frac{1}{w_i}\left\{(1-\alpha)\left[L_j w_j D_{ji}(w) - L_i w_i D_{ij}\right]\right\}
\end{aligned}$$

And because trade balance condition: $L_i p_{mi} q_i D_{ij} = L_j p_{mj} q_j D_{ji}$, so $L_j w_j D_{ji}(w) - L_i w_i D_{ij} = 0$. Then $Z(w)$ the labor excess demand is zero.

3.4.4 Trade Share Value

From the closed economy section, we have already got the wages are in both countries :

$$\begin{aligned}
\text{country } i &: \quad w_i = \alpha^\alpha (1-\alpha)^{1-\alpha} \lambda_i^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}} \\
\text{country } j &: \quad w_j = \alpha^\alpha (1-\alpha)^{1-\alpha} \lambda_j^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}}
\end{aligned}$$

And because (31and32), we get:

$$
\begin{aligned}
D_{ij} &= \frac{\psi_{ij}}{\psi_{ii}+\psi_{ij}} = \frac{w_j^{\beta} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}}{w_j^{\beta} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}} + w_i^{\beta} P_{mi}^{1-\beta}\frac{1}{\tau_{ii}}} = \frac{\lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}}{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ii}} + \lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}} \\
D_{ji} &= \frac{\psi_{ji}}{\psi_{ji}+\psi_{jj}} = \frac{w_i^{\beta} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}}}{w_i^{\beta} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}} + w_j^{\beta} P_{mj}^{1-\beta}\frac{1}{\tau_{jj}}} = \frac{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}}}{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}} + \lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{jj}}}
\end{aligned}
$$

further simplication I get:

$$
D_{ij} = \frac{\lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}}{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ii}} + \lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}} \tag{42}
$$

$$
D_{ji} = \frac{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}}}{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}} + \lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{jj}}} \tag{43}
$$

Also from closed economy section:

$$
\begin{aligned}
P_{mi} &= \alpha^{\alpha}(1-\alpha)^{1-\alpha}(AB)^{\frac{\alpha}{\beta}}\lambda_i^{-\frac{\theta\alpha}{\beta}} \\
P_{mj} &= \alpha^{\alpha}(1-\alpha)^{1-\alpha}(AB)^{\frac{\alpha}{\beta}}\lambda_j^{-\frac{\theta\alpha}{\beta}}
\end{aligned}
$$

Using this we can simplified the shares of those two countries:

$$
\begin{aligned}
D_{ij} &= \frac{\lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}}{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ii}} + \lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{ij}}} = \frac{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}} + \frac{1}{\tau_{ii}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}} \\
D_{ji} &= \frac{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}}}{\lambda_i^{\theta(1-\alpha)} P_{mi}^{1-\beta}\frac{1}{\tau_{ji}} + \lambda_j^{\theta(1-\alpha)} P_{mj}^{1-\beta}\frac{1}{\tau_{jj}}} = \frac{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}} + \frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}
\end{aligned}
$$

3.5 Effects of Technology Changes

Assume that country i has gone through a productivity revolution, then country $i's$ the productivity levelit would be improved. It means that both productiviey absolute advantage and technological heterogeneity increase. But I have mentioned in this two-country case, I assume θ is identical in both countries. So I neglect the consequence caused by changing θ, only focus on the results caused by larger λ_i. For easier I assume that the country j 's productivity level is stable and does not change[16].

[16]The proofment is given in Appendix 5.3

The conclusions which are caused by the country $i's$ technology changes are following.

Share D_{ij}'s change is:

$$D_{ij}=\frac{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}+\frac{1}{\tau_{ii}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}}$$

We can see from the function, D_{ij} is negative correlated with λ_i when $\beta>\alpha$. This means that the volume of country j's exporting to country i decreases with larger λ_i, if the final goods are more labor intensive than intermediate goods. Vice verse, if final good production is less labor intensive than intermediate good production, $\beta<\alpha$, D_{ij} is positive correlated with λ_i.[17] The volume of country j's exporting to country i will increase when λ_i becomes bigger.

Share D_{ji}'s change is:

$$D_{ji}=\frac{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}+\frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}$$

D_{ji} will increase when the λ_i gets improvement, when $\alpha<\beta$. D_{ji} will decrease when the λ_i gets improvement, when $\alpha>\beta$.The volume of country i's exporting to country j increases with increasing λ_i when α is smaller than β. It means country i can enlarge its exportation to country j by improving productivity level under the condition that intermediate goods production is more labor intensive than final goods production[18].

In summary, as country i's productivity level increases (λ_i gets larger), D_{ji} and D_{ij}'s changes direction also depend on the Cob-Douglas parameters α and β. The exporting volume of country i to country j will increase when intermediate goods production is more labor intensive than final goods production. On the contrary the change of importing volume from country j will decrease when the intermediate good is more labor intensive than final good with increasing λ_i. If the final goods production is more labor intensive than the intermediate goods production, importing volume of country i from country j will increase with increasing λ_i.

It is same that I have shown in Section 2, country i 's wage level will increase with increasing λ_i, while country j 's wage level does not chage.

Country i 's GDP is

$$GDP=L_iw_i=\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}\lambda_i^{\frac{\theta(1-\alpha)}{\beta}}\left(AB\right)^{\frac{\alpha-1}{\beta}}$$

[17] Check the proofment in Appendix 5.7.1

[18] This result is shown in Appendix 5.7.2

Obviously, GDP is determinded by the labor supply and wage level. When λ_i gets improvement GDP will be enlarged[19].If country i experiences productivity improving, its GDP will accordingly be improved.

Science there is nothing happened on country j's productivity level, country j' GDP will not change.

Country i 's real consumption is:

$$\frac{GDP}{L_i p_i c_i} = A\left(AB\right)^{-\frac{\alpha}{\beta}} \lambda_i^{\frac{\theta(\alpha-\beta)}{\beta}} \alpha^{-\alpha}(1-\alpha)^{\alpha-1}$$

I find out that there exist two cases:

(1) If $\alpha < \beta$, λ_i improves, the real consumption will increase.

(2) If $\alpha > \beta$, λ_i improves, the real consumption will decrease.

We can see that the country i's real consumption will be improved when $\alpha < \beta$ and the country i's real consumption will be cutted when $\alpha > \beta$, with technological revolution happening.

And because λ_j has no change, country j 's real consumption does not chage[20].

Above analysis tells the story that λ is a very improtant factor to affect many variable in an economy. The Cobb-Douglas parameters α and β also play important roles here. In this model, when country i experiences productivity improvement, its GDP will get enlarged. The real consumption of country i increases with larger λ_i if the intermediate good is more labor intensive than fianl good. The real consumption of country i decreases with bigger λ_i if the final good is more labor intensive than the intermediate good.

[19]The evidence is in Appendix 5.7.3

[20]The details of proofment are in Appendix 5.7.4

4 Conclusions

This paper proposes a simplified version of the Alvarez and Lucas (2007) trade theory. I follow up Alvarez and Lucas (2007)'s analysis by transferring it to the two-country case to study how the productivity level affects trade and other economic variables in a perfect-competition and general equilibrium model. The equilibrium wage is a function of the productivity level which affects trade and other economic variables in a general equilibrium model. In particular, I study how the changes of λ and θ affect trade and economy.

My conclusions in this paper are as follows.

The absolute productivity advantage improvement with the fixed technology heterogeneity, increases the wage level, intermediate goods production, final goods production and total production, but decreases the price index of intermediate goods under autarky economy. As the technological absolute advantage (mainly reflected by λ[21]) increases, the wage level will increase accordingly, and the real wage level will also increase. At the same time the tradables (intermediate goods) price index will decrease. The country can benefit from technological progress, and improvement in productivity creates a price difference which again creates a platform for bilateral trade.

The appearance of price difference gives possibility of trade in the world. The technology heterogeneity (mainly reflected by θ) plays very important role in this model, but how the change of θ affects variables in this economy can not be determined.

Afterwards, I studied a two-country version that is a small, open economy which includes two countries i and j trading with each other. I assume that only country i goes through a technological revolution to get technology improvement and then study the consequences of that. In this paper I only study the improvement of the productivity absolute advantage and omit the change of technological heterogeneity. Country i's GDP will increase along with the upgrading absolute advantage level. Going through technology improvement can enlarge the country i's trade volume compared to country j, when the intermediate goods production is more labor intensive than the final goods production, in other words, country i will experience export expansion relative to country j.It means that in the world market country i's goods will be more competitive in the world market under the special labor intensive condition. At the same time, country i's wage level get improved and would be higher than that of country j. In the case of full mobility of labor between countries, more labor will move to country i and gradually, the wage levels of the two countries will reach a new but equal level.

Implication of changing θ, the parameter mainly describes technological heterogene-

[21]The details are given in Appendix B.

ity, is only shortly discussed in this paper. However, the productivity parameter θ is a important issue in the theory of trade.

References

[1] Janathan Eaton and Samuel Kortum (2005) "Technology in the Global Economy: A Framework for Quantitative Analysis"

[2] Janathan Eaton and Samuel Kortum (2002) "Technology, Geography and Trade" Econometrica, Vol 70, No.5 (Sep.,2002) pp. 1741-1779

[3] Janathan Eaton and Samuel Kortum (1999) "International Technology Diffusion: Theory and Measurement" International Economic Review.40 pp537-570

[4] Alvarez Fernando and Robert E. Lucas, Jr (2007) "General Equilibrium Analysis of the Eaton-Kortum Model" NBER Working paper # 11764

[5] Broda Christian and David E. Weinstein (2005) "Globalization and the Gains from Variety"

[6] Baxter Marianne (1992) "Fiscal Policy, Specialization and Trade in the Two-Sector Model: The Return of Ricardo" Journal of Political Economy,100, pp 713-744

[7] Ethier Wilfred J. (1982) "National and International Returns to Scale in the Modern Theory of International Trade" American Economic Review, 72, pp 389-405

[8] Michael E. Waugh (2009) "Internationa Trade and Income Difference", Federal Reserve Bank of Minneapolis Research Department Staff Report #435

[9] Thomas Chaney (2006) "Distorted Gravity: Heterogeneous Firms, Market Structure and The Geography of International Trade"

[10] Natalia Ramondo (2006) "Size, Geography and Multinational Production" Job Market paper (Jan., 2006)

[11] Hall Robert E. and Charles I. Jones (1999) "Why Do Some Countries Produce So Much More Output Per Worker Than Other Countries" Quarterly Journal of Economics, 114, pp 83-115

[12] Helpman Elhanan (1981) "International Trade in the Presence of Product Differentiation, Economies of Scale and Monopolistic Competition" Journal of International Economics, 11 pp 305-340

[13] Helpman Elhanan and Paul R. Krugman (1985) "Market Structure and Foreign Trade" Cambridge: MIT Press

[14] Helpman Elhanan (1989) "Have International Transportation Costs Declined?" University of Chicago working paper

[15] Helpman Elhanan (1989) "Have International Transportation Costs Declined?" University of Chicago working paper

[16] Krugman Paul (1979) "Increasing Returns, Monopolistic Competition and International Trade" Journal of International Econoics, 9, pp 469-480

[17] Wilson Charles A. (1909) "On the General Structure of Ricardian Models with a Coninuum of Goods: Applications to Growth, Tariff Theory and Technical Changes" Econometrica 48, pp 1675-1702

5 Appendix

5.1 Frechet Distribution

Here let $X(x_{1,}x_2, ...x_n)$ be random variable vector independent across goods, the cdf is:

$$F_X(x) = 1 - e^{-\lambda x}, X \geq 0$$

This means that the cost draw $X = (x_1, x_2, ..., x_n)$ follows exponential distribution. Now let us introduce the a theorem of the cdf.

Let X have cdf $F_X(x)$, let $Y = g(X)$, and X, Y are defined in the $[0, \infty)$. Then we will have :

(a) If g is an increasing function on $[0, \infty)$, $F_Y(y) = F_X(g^{-1}(y))$;

(b) If g is an decreasing function on $[0, \infty)$ and X is a contimuous random variable, $F_Y(y) = 1 - F_X(g^{-1}(y))$;

And we have the function:

$$Y = X^{-\theta}$$

because of the relationship between X and Y :

$$\frac{dY}{dX} = -\theta X^{-\theta-1}$$

Since $X \geq 0, X^{-\theta-1} \geq 0$, then

$$\frac{dY}{dX} = -\theta X^{-\theta-1} \leq 0$$

it means that Y is decreasing in X.What's more the inverse function of z is:

$$x = y^{-\frac{1}{\theta}}$$

So from the theorem, we can get the cdf of Y :

$$\begin{aligned} F_Y(y) &= \Pr(y_n \leq y) = 1 - F_X\left(y^{-\frac{1}{\theta}}\right) \\ &= 1 - \left[1 - e^{-\lambda y^{-\frac{1}{\theta}}}\right] \\ &= e^{-\lambda y^{-\frac{1}{\theta}}} \end{aligned}$$

So we have

$$F_Y(y) = \Pr(y_n \leq y) = e^{-\lambda y^{-\frac{1}{\theta}}}$$

And the Frechet Distribution is defined:

$$F_n(Z \leq z) = \exp\left(-sz^{-\alpha}\right)$$

So then we can say that Y follows Frechet distribution with the parameter λ and $\frac{1}{\theta}$.

From Eaton-Kortum model, we know that λ and $\frac{1}{\theta}$ governs the mean and varience of the intermediate goods' productivity (we think them as random variables). And also λ is the parameter that decides the exact position of the probability density and $\frac{1}{\theta}$ is the parameter that decides the shape of the probability density function.

5.2 Mean, Variance of Frechet Distribution

Here we try to find the expectation value and variance of random variable $y(y = x^{-\theta})$.

We have the cdf of y :

$$F_Y(y) = \Pr(y_n \leq y) = e^{-\lambda y^{-\frac{1}{\theta}}}, y \in [0, \infty)$$

So we can get the pdf of y :

$$f_Y(y) = \frac{\lambda}{\theta} y^{-1-\frac{1}{\theta}} e^{-\lambda y^{-\frac{1}{\theta}}}, y \in [0, \infty)$$

Then we the expectation value is:

$$\begin{aligned} E(Y) &= \int_0^\infty y f_Y(y)\, dy \\ &= \int_0^\infty y \frac{\lambda}{\theta} y^{-1-\frac{1}{\theta}} e^{-\lambda y^{-\frac{1}{\theta}}} dy \\ &= \int_0^\infty \frac{\lambda}{\theta} y^{-\frac{1}{\theta}} e^{-\lambda y^{-\frac{1}{\theta}}} dy \end{aligned}$$

Let $u = -\lambda y^{-\frac{1}{\theta}}$, then $\frac{du}{dy} = \frac{\lambda}{\theta} y^{-\frac{1}{\theta}-1} \Rightarrow du = \frac{\lambda}{\theta} y^{-\frac{1}{\theta}-1} dy$ and $y = \left(-\frac{u}{\lambda}\right)^{-\theta} = \lambda^\theta (-u)^{-\theta}$

$$E(Y) = \lambda^\theta \int_0^\infty (-u)^{-\theta} u e^{-u} du = \lambda^\theta \int_0^\infty u^{1-\theta} e^{-u} du$$

And the Gamma Function is : $\Gamma(\xi) = \int_0^\infty x^{\xi-1} e^{-x} dx$, here $\xi = 1 - \theta$

$\Rightarrow \lambda^{\theta} \int_0^{\infty} u^{1-\theta} e^{-u} du = \lambda^{\theta} \Gamma(1-\theta)$

So the expectation value of random variable $Y \left(= X^{-\theta}\right)$:

$$E(Y) = \lambda^{\theta} \Gamma(1-\theta) \Rightarrow E\left(X^{-\theta}\right) = \lambda^{\theta} \Gamma(1-\theta)$$

Now we start to show how to get the variance.

First, we should introduce a theorem about the expectation value and variance.

Theorem 4 *The variance of X, if it exists, may also be calculated as follows:*

$$Var(X) = E\left(X^2\right) - \left[E(X)^2\right]$$

Then if we find $E(Y^2)$, it is easy to get the variance:

$$\begin{aligned} E\left(Y^2\right) &= \int_0^{\infty} y^2 f_Y(y)\, dy \\ &= \int_0^{\infty} y^2 \frac{\lambda}{\theta} y^{-1-\frac{1}{\theta}} e^{-\lambda y^{-\frac{1}{\theta}}} dy \end{aligned}$$

The same as above, it is easy to get:

$$E\left(Y^2\right) = \lambda^{2\theta} \Gamma(1-2\theta)$$

And

$$\left[E(Y)^2\right] = \left[\lambda^{\theta} \Gamma\left(1 - \frac{1}{\theta}\right)\right]^2$$

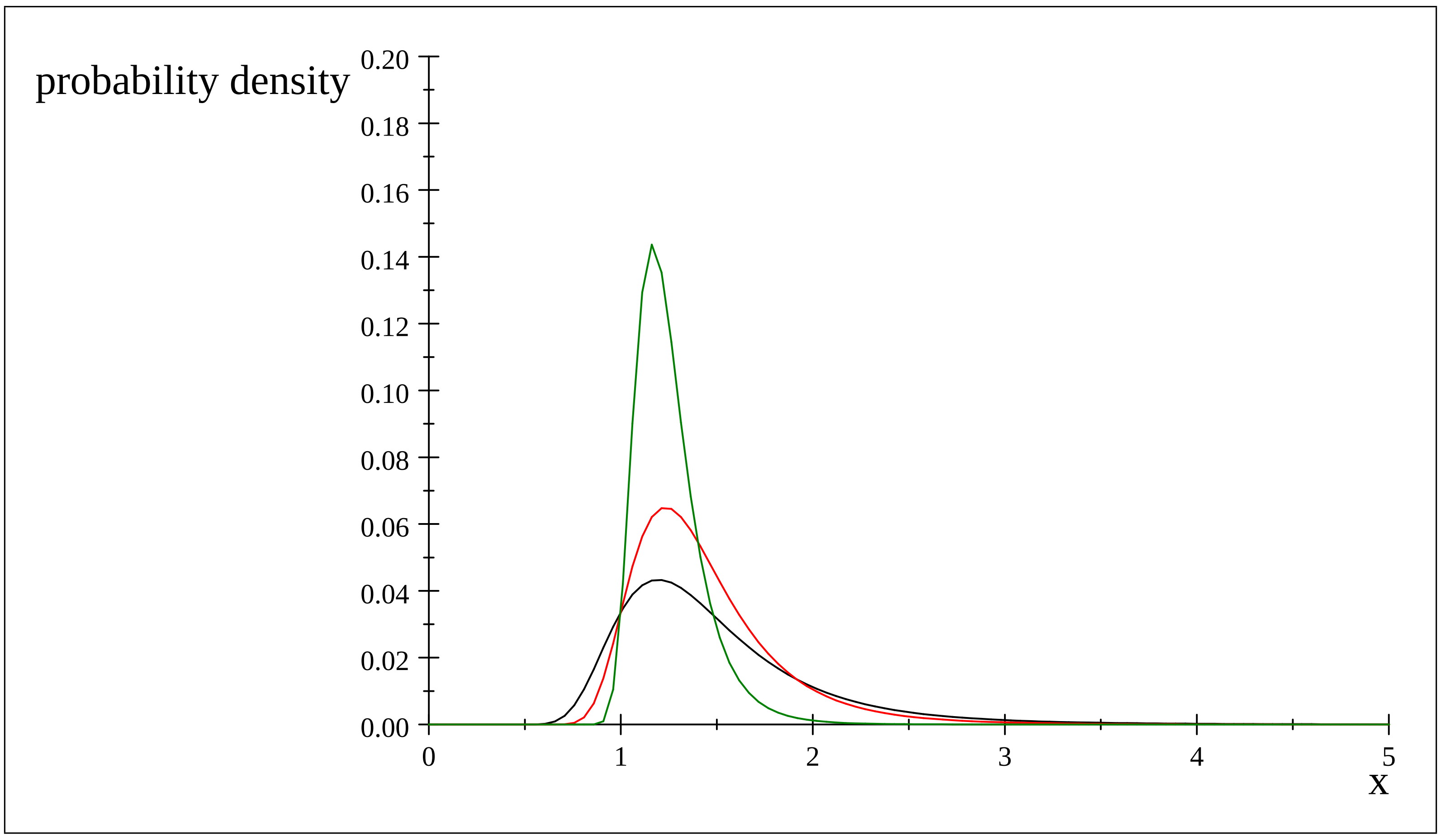
So the variance should be:

$$\begin{aligned} Var(Y) &= E\left(Y^2\right) - \left[E(Y)^2\right] \\ &= \lambda^{2\theta} \Gamma\left(1 - \frac{2}{\theta}\right) - \lambda^{2\theta} \left[\Gamma\left(1 - \frac{1}{\theta}\right)\right]^2 \\ &= \lambda^{2\theta} \left\{\Gamma\left(1 - \frac{2}{\theta}\right) - \left[\Gamma\left(1 - \frac{1}{\theta}\right)\right]^2\right\} \end{aligned}$$

5.3 Effects of Changing λ and θ

Now we talk more about the effects of parameters (λ and θ) of this distribution.

First, we let λ fixed at some value and see what will change if θ value changes.

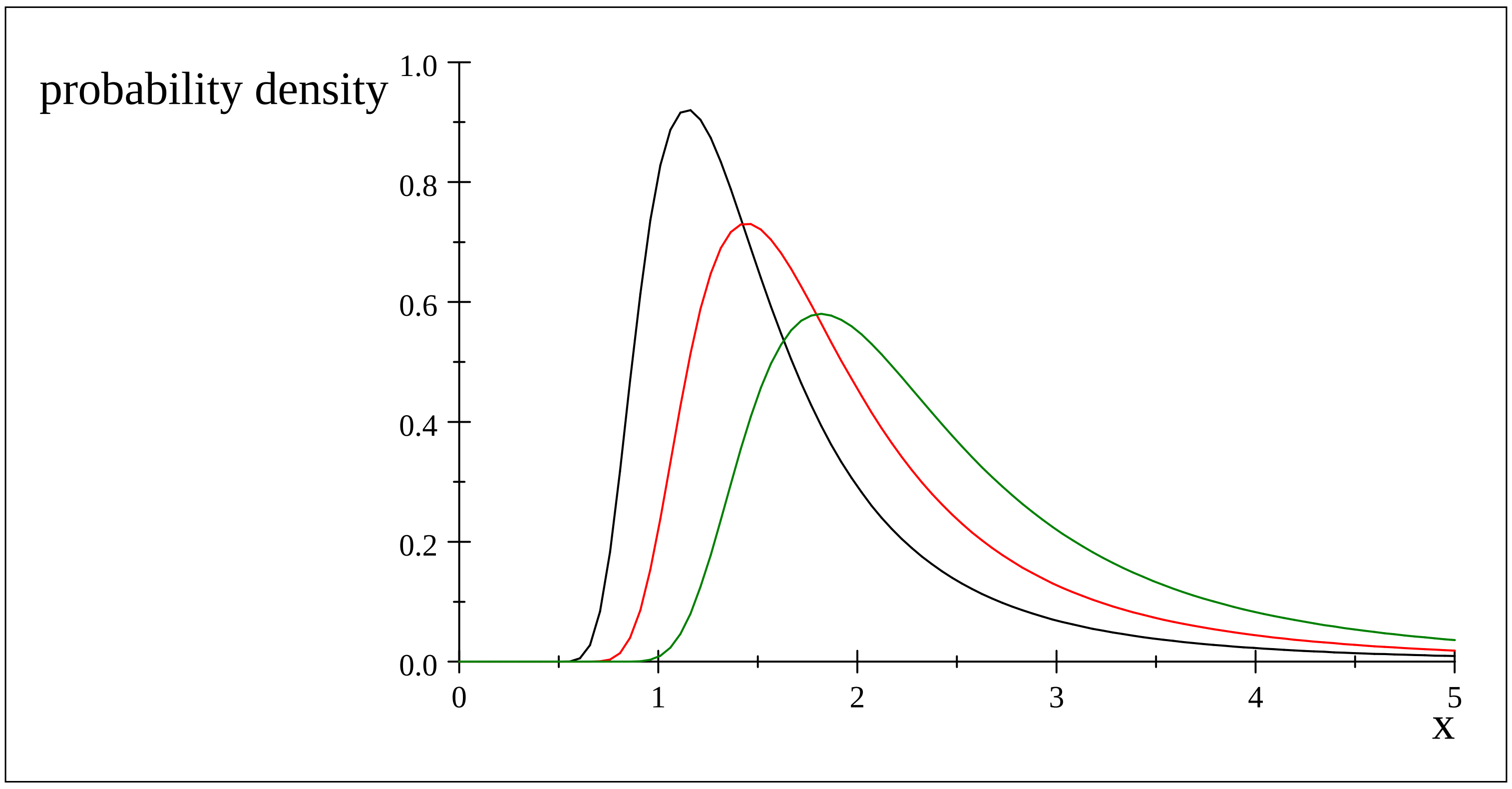


In the above figure: when $\lambda = 5$

the black line: $\theta = \frac{1}{2}$; the red line: $\theta = \frac{1}{3}$; the green line: $\theta = \frac{1}{4}$. We can see that when θ changes the value, the variance of that distribution will accordingly change very much.

It can be seen that the shape of the distribution figure has chaged with growing θ. When θ gets smaller and smaller value the highest point will be higher and higher. If we take the productivities as iid randoms here, we also can conclude that the technological heteroeneity improves with decreasing θ, but improve with increasing $\frac{1}{\theta}$.

Change the λ value to see what will happen, when let θ be some fixed value, three values of dispersion λ.



In the above figure:

the black line: $\lambda = 2$; the red line: $\lambda = 4$; the green line: $\lambda = 8$

We can observe that if λ gets larger, the position of distribution figure will move to the right hand. This means that there is a larger mean value in these iid randoms when λ get larger. In our model, we can say that the technological random variables' mean increases with growing λ, in the other side it is the technological absolute advantage get improved when λ get improvement.

5.4 Solve the Equilibrium Prices of Closed Economy: Equilibrium prices of final goods

Cost minimizing behavior of all producers will satisfy :

$$w_f s_f + p_m q_f \quad \text{subject to } y = c = s_f^{\alpha} q_f^{1-\alpha}$$

w_f is the wage rate

p_m is the price of intermediate goods

Because in one country the wage rates are same :$w_f = w_m$

(1) Solving the constraint for q_f ,we see that this problem is equivalent to:$\min_{s_f} w_f s_f + p_m y^{\frac{1}{1-\alpha}} s_f^{\frac{-\alpha}{1-\alpha}}$

The first-order condition is $w_f - \frac{\alpha}{1-\alpha} p_m y^{\frac{1}{1-\alpha}} \; s_f^{\frac{-1}{1-\alpha}} = 0$

which gives us the conditional demand for s_f : $s_f = \left(\frac{1-\alpha}{\alpha} \frac{w_f}{p_m} \right)^{\alpha - 1} y$

(2) As the same solving the constraint for we see that this problem is equivalent to:min $w_f y^{\frac{1}{\alpha}} q_m^{\frac{\alpha-1}{\alpha}} + p_m q_m$

The first-order condition is $p_m + w_f y^{\alpha} \frac{\alpha - 1}{\alpha} q_m^{\frac{-1}{\alpha}}$

which gives us the conditional demand for q_m : $q_m = \left(\frac{\alpha}{1-\alpha} \frac{p_m}{w_f} \right)^{-\alpha} y$

The cost (marginal cost) function is :

$w_f \left(\frac{1-\alpha}{\alpha} \frac{w_f}{p_m} \right)^{\alpha-1} y + p_m \left(\frac{\alpha}{1-\alpha} \frac{p_m}{w_f} \right)^{-\alpha} y = y$

$\Longrightarrow \left[\left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} + \left(\frac{\alpha}{1-\alpha} \right)^{-\alpha} \right]^{-1} w_f^{-\alpha} p_m^{1-\alpha}$

and we have :

$$\begin{aligned} \left[\left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} + \left(\frac{\alpha}{1-\alpha} \right)^{-\alpha} \right]^{-1} &= \left(\frac{\alpha}{1-\alpha} \right)^{-\alpha} \left[\left(\frac{\alpha}{1-\alpha} \right) + 1 \right] = \\ \left(\frac{\alpha}{1-\alpha} \right)^{-\alpha} \left(\frac{1}{1-\alpha} \right) &= \alpha^{-\alpha} (1-\alpha)^{\alpha-1} \end{aligned}$$

So we get

$$p_m = \alpha^{-\alpha} (1-\alpha)^{\alpha-1} w_f^{-\alpha} p_m^{1-\alpha} \tag{1.1}$$

So this is the price of final goods under closed condition.

5.5 Solve the Equilibrium Prices of Closed Economy: Equilibrium prices of intermediate goods

The intermediate goods' production function:

$$q_f(x) = \left[\int_0^\infty q_m(x)^{1-\frac{1}{\eta}}\phi(x)dx\right]^{\frac{\eta}{\eta-1}}$$

We have been told above that there is only labor input expect for intermediate goods during the production.

So the labor endowment is allocated over final goods production: s_f

And we can get the labor to produce intermediates should be $s(x)$, $x > 0$,then the labor supply to produce intermediate goods is $\int_0^\infty s\left(x\right)\phi\left(x\right)dx$.

So we can get the labor supply:

$$s_f + \int_0^\infty s\left(x\right)\phi\left(x\right)dx \leq 1$$

So the total production function is :

$$q_f + \left[\int_0^\infty q_m(x)^{1-\frac{1}{\eta}}\phi(x)dx\right]^{\frac{\eta}{\eta-1}} \leq q$$

$$\Longrightarrow q = \left[\int_0^\infty q(x)^{1-\frac{1}{\eta}}\phi(x)dx\right]^{\frac{\eta}{\eta-1}}$$

Following E-K model (2002) and assuming that the density ϕ is exponential with parameter $\lambda : x \backsim \exp(\lambda)$.These x draws are then amplified in percentage terms by the parameter θ.

Here the random variables $x^{-\theta}$ then have a Frechet distribution.

So the intermediate goods producrr $x's$ production function is :

$$q_f\left(x\right) = x^{-\theta}s\left(x\right)^{\beta}q_m\left(x\right)^{1-\beta}$$

$q_m\left(x\right)$ is looked as the input bundle to produce intermediate goods

$s\left(x\right)$ is the allocated labor in intermediate goods production

Then we should try to find the intermediate goods price p_m.There are two ways which will give the solution.

Now minimize the cost of producing intermediate goods"

$$\min ws(x)+p_m q_m(x) \text{ subject to } q(x)=x^{-\theta}s(x)^{\beta}q_m(x)^{1-\beta}$$

w is the wage rate ,in the same country we can take it as granted that $w=w_f$

(1) Solving the constraint for $q_m(x)$,we see that this problem is equivalent to:$\min_{s(x)} ws(x)+p_m q(x)^{\frac{1}{1-\beta}}x^{\frac{\theta}{1-\beta}}s(x)^{\frac{-\beta}{1-\beta}}$

The first-order condition is $w-\frac{\beta}{1-\beta}p_m\ q(x)^{\frac{1}{1-\beta}}s(x)^{\frac{-1}{1-\beta}}=0$

which gives us the conditional demand for $s(x): s(x)=\left(\frac{1-\beta}{\beta}\frac{w}{p_m}\right)^{\beta-1}q(x)$

(2) As the same solving the constraint for we see that this problem is equivalent to:$\min wq(x)^{\frac{1}{\beta}}q_m(x)^{\frac{\beta-1}{\beta}}+p_m q_m(x)$

The first-order condition is $p_m+\frac{\beta-1}{\beta}wq(x)^{\frac{1}{\beta}}q_m(x)^{\frac{-1}{\beta}}$

which gives us the conditional demand for $q_m(x): q_m(x)=\left(\frac{\beta}{1-\beta}\frac{p_m}{w}\right)^{-\beta}q(x)$

The cost (marginal cost) function is : $w\left(\frac{1-\beta}{\beta}\frac{w}{p_m}\right)^{\beta-1}q(x)x^{\theta}+p_m\left(\frac{\beta}{1-\beta}\frac{p_m}{w}\right)^{-\beta}q(x)x^{\theta}=q(x)$

$\Longrightarrow\left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta}+\left(\frac{\beta}{1-\beta}\right)^{-\beta}\right]^{-1}x^{\theta}w_f^{-\beta}p_m^{1-\beta}=p_m$

and we have :$\left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta}+\left(\frac{\beta}{1-\beta}\right)^{-\beta}\right]=\beta^{-\beta}(1-\beta)^{\beta-1}$

So we get

$$p(x)=\beta^{-\beta}(1-\beta)^{\beta-1}w^{-\beta}p_m^{1-\beta}x^{\theta}$$

This is the price of intermediate goods under closed condition.

5.6 Analysis of relationship between λ and economical variables in closed economy

5.6.1 Wage rate:

The wage rate is:

$$w=\alpha^{\alpha}(1-\alpha)^{1-\alpha}\lambda^{\frac{\theta(1-\alpha)}{\beta}}(AB)^{\frac{\alpha-1}{\beta}}$$

we can have

$$\frac{dw}{d\lambda}=\frac{\theta(1-\alpha)}{\beta}\alpha^{\alpha}(1-\alpha)^{1-\alpha}\lambda^{\frac{\theta(1-\alpha)-\beta}{\beta}}(AB)^{\frac{\alpha-1}{\beta}}$$

It is very easy to see that $\frac{dw}{d\lambda}>0$, so we can conclude that w is increasing in λ.

5.6.2 The individual intermediate good price:

The individual price of intermediate goods

$$p\left(x\right)=\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}\lambda^{\frac{\theta(\beta-\alpha)}{\beta}}\left(AB\right)^{\frac{\alpha}{\beta}}A^{-1}$$

Take the first-order condition:

$$\frac{dp\left(x\right)}{d\lambda}=\frac{\theta\left(\beta-\alpha\right)}{\beta}\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}\lambda^{\frac{\theta(\beta-\alpha)-\beta}{\beta}}\left(AB\right)^{\frac{\alpha}{\beta}}A^{-1}$$

As we have know, $0<\alpha<10$, $\beta>0$, $A>0, B>0$, so that $\frac{dp(x)}{d\lambda}$ is larger or smaller than zero depens $\left(\beta-\alpha\right)$.

If $\beta>\alpha$, $\beta-\alpha>0$, then $\frac{dp(x)}{d\lambda}>0$.

If $\beta<\alpha$, $\beta-\alpha<0$, then $\frac{dp(x)}{d\lambda}<0$.

5.6.3 The intermediate goods' price index

The intermediate goods' price index is:

$$P_m=\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}\left(AB\right)^{\frac{\alpha}{\beta}}\lambda^{-\frac{\theta\alpha}{\beta}}$$

And

$$\frac{dP_m}{d\lambda}=\left(-\frac{\theta\alpha}{\beta}\right)\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}\left(AB\right)^{\frac{\alpha}{\beta}}\lambda^{-\left(\frac{\theta\alpha+1}{\beta}\right)}$$

It is obvious that $\frac{dP_m}{d\lambda}<0$, so the intermediate goods' price index is decreasing in λ.

5.6.4 The final good's price

The price of final good is:

$$p=\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}w_f^{\alpha}P_m^{1-\alpha}$$

It is obvious that :

$$\frac{dp}{dP_m}=\left(1-\alpha\right)\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}w_f^{\alpha}P_m^{-\alpha}$$

We can say that p is an increasing function of P_m, and from above we already shown that $\frac{dP_m}{d\lambda}<0$. Using "Chain Rule", easy to show that p is negative relacated with λ.

5.6.5 The intermediate goods used to produce final goods

The intermediate goods used to produce final goods is:

$$q_f = (1-\alpha)\,\lambda^{\frac{\theta}{\beta}}\,(AB)^{-\frac{1}{\beta}} \Rightarrow \frac{dq_f}{d\lambda} = \frac{\theta}{\beta}\,(1-\alpha)\,\lambda^{\frac{\theta}{\beta}-1}\,(AB)^{-\frac{1}{\beta}}$$

Of course $\frac{dq_f}{d\lambda} > 0$, the it is natural that q_f increases with larger λ.

5.6.6 The intermediate input bundle to produce intermediate goods

The intermediate input bundle to produce intermediate goods is

$$\begin{aligned} q_m(x) &= (1-\alpha)\left[\left(\frac{1-\beta}{\beta}\right)(AB)^{-\frac{1}{\beta}}\,\lambda^{\frac{\theta}{\beta}}\right] \\ &\Rightarrow \frac{dq_m(x)}{d\lambda} = \frac{\theta}{\beta}\,(1-\alpha)\left[\left(\frac{1-\beta}{\beta}\right)(AB)^{-\frac{1}{\beta}}\,\lambda^{\frac{\theta}{\beta}-1}\right] \end{aligned}$$

Because of $\frac{dq_m(x)}{d\lambda} > 0$, then $q_m(x)$ is an increasing function of λ.

5.6.7 Total production of intermediate

Total production of intermediate is:

$$q = (AB)^{-\frac{1}{\beta}}\,\lambda^{\frac{\theta}{\beta}\left(\frac{1-\alpha}{\beta}\right)}$$

And

$$\frac{dq}{d\lambda} = \frac{\theta}{\beta}\,(AB)^{-\frac{1}{\beta}}\,\lambda^{\frac{\theta}{\beta}-(\theta+\beta)}$$

Easy to see that $\frac{dq}{d\lambda} > 0$, so the total production of intermediate goods of this economy is increasing with λ increasing.

5.7 Analysis of relationship between λ and economical variables in open economy

5.7.1 D_{ij}'s change:

We have the function of D_{ij}

$$
\begin{aligned}
D_{ij} &= \frac{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}+\frac{1}{\tau_{ii}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}} \\
\Rightarrow \frac{dD_{ij}}{d\lambda_i} &= -\left[\frac{1}{\tau_{ji}}\frac{\theta(\beta-\alpha)}{\beta}\right]\frac{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ij}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}+\frac{1}{\tau_{ii}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}-1}}
\end{aligned}
$$

From above, it is easy to observe:
If $\beta > \alpha$, then $\frac{dD_{ij}}{d\lambda_i} < 0$.
If $\beta < \alpha$, then $\frac{dD_{ij}}{d\lambda_i} > 0$.

5.7.2 For D_{ji}'s change:

We have D_{ji}

$$
\begin{aligned}
D_{ji} &= \frac{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}+\frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}} = 1-\frac{\frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}{\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}}+\frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}} \\
\Rightarrow \frac{dD_{ji}}{d\lambda_i} &= \frac{\frac{1}{\tau_{ji}\tau_{jj}}\frac{\theta(\beta-\alpha)}{\beta}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}-1}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}}{\left[\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}-1}+\frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}\right]^2}
\end{aligned}
$$

It is obvious that the part $\left[\frac{1}{\tau_{ji}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}-1}+\frac{1}{\tau_{jj}}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}\right]^2$ is always nonnegative, the key part which can decide the whether $\frac{dD_{ji}}{d\lambda_i}$ is positive is the part $\frac{1}{\tau_{ji}\tau_{jj}}\frac{\theta(\beta-\alpha)}{\beta}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}-1}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}$. Because $\frac{1}{\tau_{ji}\tau_{jj}}\lambda_i^{\frac{\theta(\beta-\alpha)}{\beta}-1}\lambda_j^{\frac{\theta(\beta-\alpha)}{\beta}}$ is naturally positive, then only $\frac{\theta(\beta-\alpha)}{\beta}$ does decide the change direction of D_{ji} with λ_i. So there are two conditinos existing:

(1) If $\alpha > \beta$, then $\beta - \alpha < 0$. This means that $\frac{\theta(\beta-\alpha)}{\beta} < 0$, $\frac{dD_{ji}}{d\lambda_i} < 0$, D_{ji} will be negative correlated with λ_i.

(2) If $\alpha < \beta$, then $\beta-\alpha > 0$. This means that $\frac{\theta(\beta-\alpha)}{\beta} > 0$, $\frac{dD_{ji}}{d\lambda_i} < 0$, D_{ji} will be positive correlated with λ_i.

5.7.3 Country i 's GDP

GDP of country i is:

$$GDP = L_i w_i = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda_i^{\frac{\theta(1-\alpha)}{\beta}} (AB)^{\frac{\alpha-1}{\beta}}$$

Then the first-order is

$$\frac{dGDP}{d\lambda_i} = \frac{\theta (1-\alpha)}{\beta} \alpha^{\alpha} (1-\alpha)^{1-\alpha} \lambda_i^{\frac{\theta(1-\alpha)-\beta}{\beta}} (AB)^{\frac{\alpha-1}{\beta}}$$

Easy to show that $\frac{dGDP}{d\lambda_i} > 0$, so we can conclude that GDP of country i increases with increasing λ_i.

5.7.4 Country i 's real consumption

Country i 's real consumption is:

$$\frac{GDP}{L_i p_i c_i} = A (AB)^{-\frac{\alpha}{\beta}} \lambda_i^{\frac{\theta(\alpha-\beta)}{\beta}} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}$$

Take first derivative :

$$\frac{d\frac{GDP}{L_i p_i c_i}}{d\lambda_i} = \frac{\theta (\alpha-\beta)}{\beta} A (AB)^{-\frac{\alpha}{\beta}} \lambda_i^{\frac{\theta(\alpha-\beta)}{\beta}} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}$$

Now whether $\frac{d\frac{GDP}{L_i p_i c_i}}{d\lambda_i}$ is lager than zero or not is decided by $(\alpha-\beta)$.
If $(\alpha-\beta) > 0 \Rightarrow \alpha < \beta$, $\frac{d\frac{GDP}{L_i p_i c_i}}{d\lambda_i} > 0$.
If $(\alpha-\beta) < 0 \Rightarrow \alpha > \beta$, $\frac{d\frac{GDP}{L_i p_i c_i}}{d\lambda_i} < 0$